Bonus Maths 2: Variable Bet Sizing in the Simplest Possible Game of Poker (JB)

I recently decided to read Part Three of The Mathematics of Poker (TMOP) more carefully than I did the first time around. I didn't get very far. Example 14.1 of Chapter 14 on No-Limit Bet Sizing left me scratching my head. At first, I thought that there must be something wrong with Chen and Ankenman's analysis. However, after checking it carefully, I found that there wasn't, but was still left with a feeling that there must be more that can be said about this game (The Half Street No Limit Clairvoyance Game - HSNLCG). So that you don't have to keep looking at TMOP, I'll start by describing the game, review its analysis for a fixed bet size and summarize Chen and Ankenman's arguments about the solution of the game with no limit betting. We can then try to get some more understanding by looking at a version of the game in which there are just two possible bet sizes, and Tom must choose with which size he will value bet and with which he will bluff; poor old, out of position John will have to decide which bet sizes he will call with his bluff catcher. We'll find that if Tom has a strong enough range, for big enough bets, the pure strategy Nash equilibrium is indeed for Tom to bet big with everything and for John to fold every time. We will also look at a reduced version of this game, for which there exists an equilibrium mixed strategy for small enough bet sizes, which involves Tom sometimes betting big for value and small as a bluff, and vice versa. This may or may not be more profitable for John than the pure strategy Nash equilibrium, depending on the parameters of the game. This mixed strategy corresponds more closely to my naive intuition about the game, and, with a bit of a mental stretch, may give us some insight into similar river situations in NLHE.

The Half Street Fixed Limit Clairvoyance Game (HSFLCG)

In the Half Street Fixed Limit Clairvoyance Game, John is dealt a bluff catcher and Tom is dealt a hand drawn randomly from a distribution that contains a fraction y of the nuts and $1-y$ of air. John's bluff catcher beats air but loses to the nuts. Both players know that this is how the hands are dealt. There is \$1 in the pot. John checks and Tom must decide between checking behind and betting \$$s$. Tom will always bet with the nuts, but he must decide between bluffing and giving up with his air. If Tom bets, John needs to decide whether or not to call with his bluff catcher. When $y = 1/2$, this is a subgame of the AKQ game in which John is always dealt a K and Tom is dealt either an A or a Q. In the HSFLCG, as y increases or decreases from $1/2$, Tom's range becomes stronger or weaker than it is in the AKQ game.

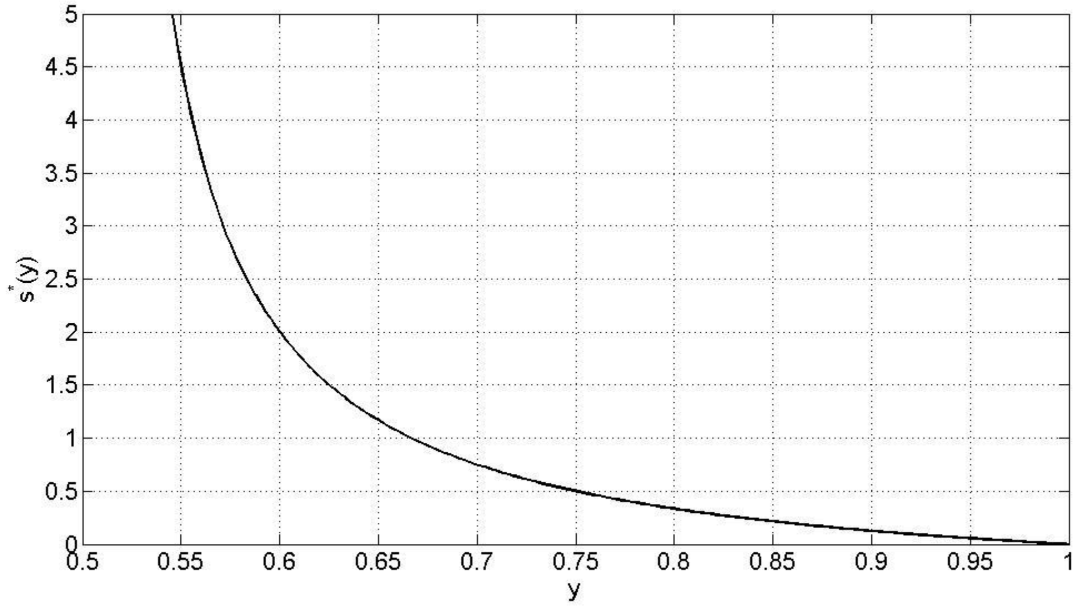
Tom's two pure strategies are Bluff and Don't Bluff, and John's two pure strategies are Call and Fold. Here's the payoff matrix for this game. A positive payoff favours Tom.

	Bluff	Don't Bluff
Call	$(2y-1)s \quad \downarrow\rightarrow$	$\downarrow \;\; ys$
Fold	$1-y$	$\leftarrow \;\; 0$

Let's start by considering what happens when $1-y > (2y-1)s$. What would John and Tom do if they started off by playing Don't Bluff/Call? Well, John would see that he loses less money by changing to the strategy Fold. Tom would then see that he could win more money by changing to the strategy Bluff. Once they hit upon the idea of playing Bluff/Fold, the shaded bottom left hand square in the matrix, they would see that neither of them can improve their payoff by changing their strategy. They have converged upon the Nash equilibrium. The arrows in the matrix show the direction in which the players have an incentive to change their strategies. In this case, Tom's range is strong enough and the bet size is large enough that poor old John just has to fold every time. The loss of $1 - y$ that he incurs by folding to all of Tom's bluffs is less than the amount he would lose by calling his (frequent and sizeable) value bets.

If you think about the condition $1 - y > (2y - 1)s$, you'll realize that it can only hold if $y > 1/2$, and $s > s^*(y)$, where

$$s^*(y) \equiv \frac{1 - y}{2y - 1}.$$



As you can see, the size of the bet that is large enough to make John fold every time gets smaller as Tom's range gets stronger (as y increases). For example, a pot-sized bet ($s^* = 1$) is big enough when $y > 2/3$, and a half pot-sized bet ($s^* = 1/2$) is big enough when $y > 3/4$.

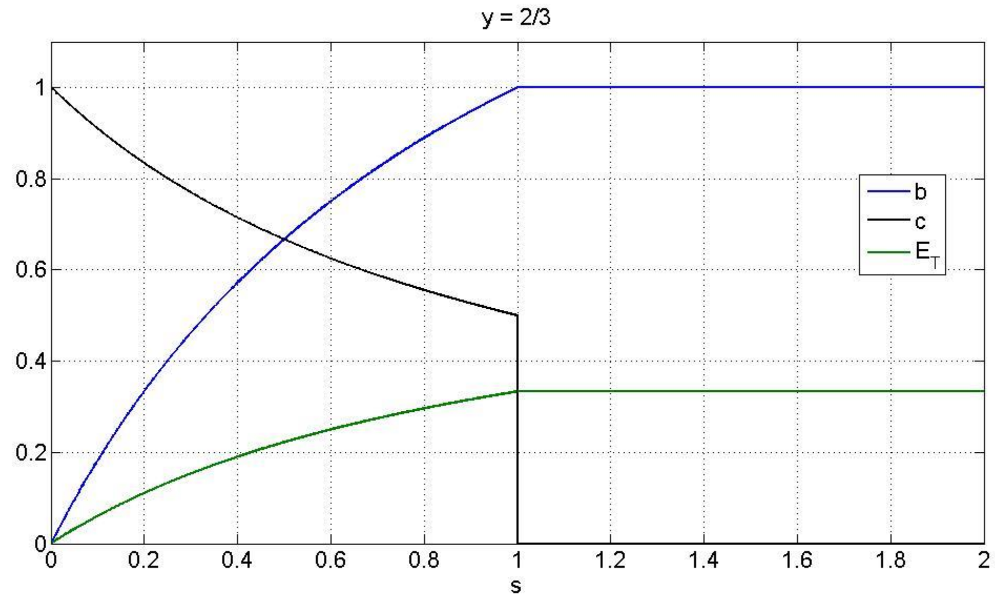
Now let's see what happens if $y < 1/2$ or $s < s^*(y)$. The payoff matrix is the same, but now $1 - y < (2y - 1)s$ and $(2y - 1)s < ys$.

	Bluff	Don't Bluff
Call	$(2y - 1)s \quad \rightarrow$	$ys \quad \downarrow$
Fold	$1 - y \quad \uparrow$	$\leftarrow \quad 0$

Now, if the players start off using Don't Bluff/Call, John will want to change to Fold, then Tom will want to change to Bluff, John will start to use Call, Tom will then use Don't Bluff, and they're back where they started. The strategies cycle around, which indicates that any equilibrium strategy must be mixed. Using either the methods given in TMOP or the minimax method that I described in my previous article, it's straightforward to show that the equilibrium strategy is

$$b = \frac{ys}{(1-y)(1+s)}, \qquad c = \frac{1}{1+s},$$

where b is Tom's bluffing frequency and c is John's calling frequency. Tom's win rate when the players use these strategies is $E_T = ys/(1+s)$. Here's a plot of both these frequencies and Tom's winrate when $y = 2/3$.



Since $s^* = 1$, John's calling frequency drops discontinuously to zero at $s = 1$ and Tom's winrate is frozen at $1/3$ for $s \geq 1$, because John isn't able to bluff catch bets that are bigger than pot-sized and just gives up to the $1 - y = 1/3$ of Tom's bets that are bluffs, as well as to his value bets.

The Half Street No Limit Clairvoyance Game (HSNLCG)

You can get most of the information in the previous section from Chapter 14 of TMOP. However, Chen and Ankenman then go on to discuss how this game plays out when $y > 1/2$ if the bet size, s, instead of being fixed, is chosen by Tom, presumably from a finite stack so that $s \leq s_{max}$. They say that (implicitly assuming that $s_{max} > s^*$), whatever bet size Tom chooses, John has to fold. In other words, even if Tom bets $0.01, John has to fold. They talk about how, if John calls, Tom can exploit him by betting small with his value hands and an amount greater than $s^*$with his bluffs. Then John can exploit this by calling big bets and folding to small bets, and from then on the argument gets a bit hazy and unconvincing in my opinion. This sort of decision comes up a lot on the river in NLHE; Tom has a polarized range and bets – is he value betting big and bluffing small or vice versa? Do we have a note on him? Is his range strong or weak? It often feels like a guessing game. Can it really be optimal for Tom to just jam in a big bet and for John to fold every time when Tom's range is strong enough? Is that how real people play? I don't think so.

Let's investigate this further by considering a game that sits somewhere between the HSFLCG and the HSNLCG.

The Half Street Big/Small Clairvoyance Game (HSBSCG)

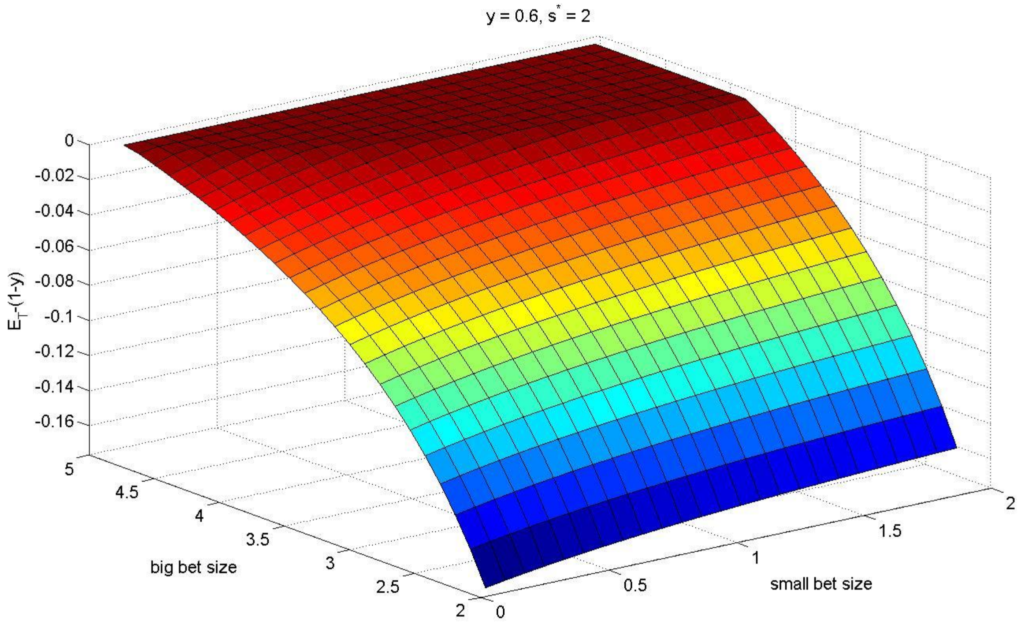
In this game, most of the rules are the same as the other Half Street Clairvoyance Games, but Tom can choose between just two bet sizes, s_1 and s_2. We will only consider the game where Tom has a strong range ($y > 1/2$) and $s_1 > s^* > s_2 > 0$. This means that Tom has at his disposal a big bet ($s_1 > s^*$, so that, in isolation, it would be big enough to make John fold every hand) and a small bet ($s_2 < s^*$, so that, in isolation, it would be small enough that John should call with a fraction $1/(1 + s_2)$ of his hands). What strategies can John and Tom use in this game? With his value hands, Tom can choose between the two bet sizings. With his air, Tom can choose between using either of the two bluff bet sizings and checking behind. He therefore has 2 x 3 = 6 strategies, which I will label $v_i b_j$ for value betting with bet size s_i and bluffing with bet size s_j, and $v_i k$ for value betting with bet size s_i and checking behind with air. John has to decide whether to call or fold when faced with each of the two bet sizes, so he has 4 strategies available to him, namely c_1c_2, c_1f_2, f_1c_2, f_1f_2, a notation that I hope is self-explanatory. Here's the payoff matrix for this game.

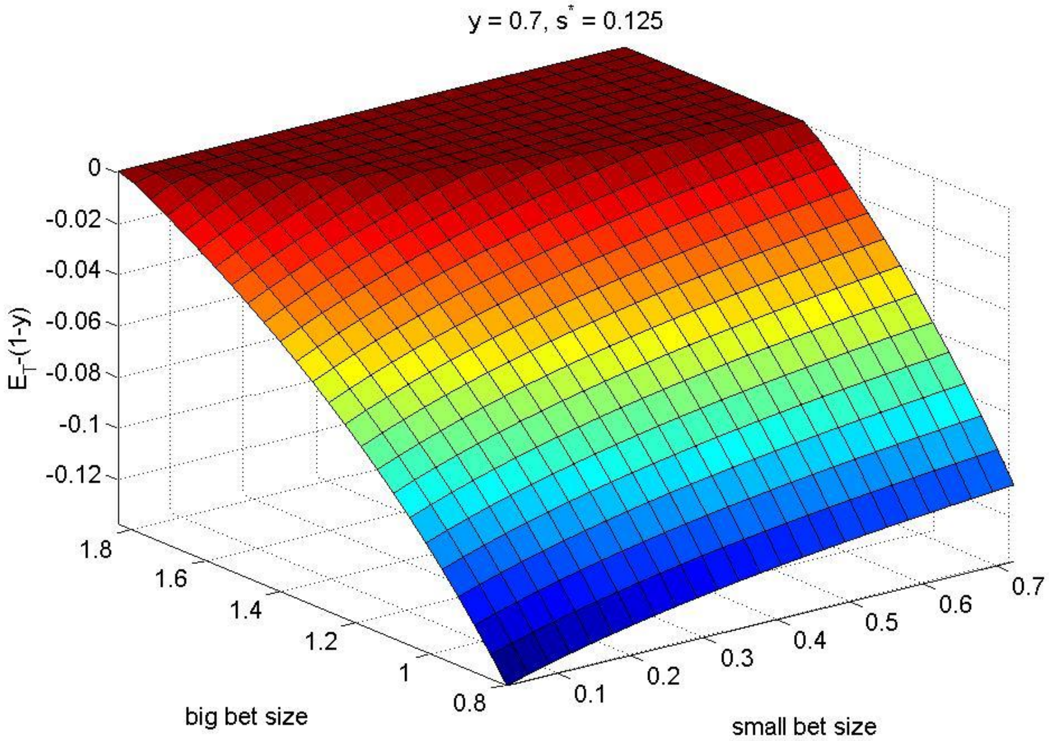
	v_1b_1	v_1b_2	v_2b_1	v_2b_2	v_1k	v_2k
c_1c_2	$(2y-1)s_1$	$ys_1-(1-y)s_2$	$ys_2-(1-y)s_1$	$(2y-1)s_2$	ys_1	ys_2
c_1f_2	$(2y-1)s_1$	ys_1+1-y $\downarrow$	$\leftarrow$ $-(1-y)s_1$	$1-y$	ys_1	0
f_1c_2	$1-y$	$-(1-y)s_2$ $\rightarrow$	$\uparrow$ ys_2+1-y	$(2y-1)s_2$	0	ys_2
f_1f_2	$1-y$	$1-y$	$1-y$	$1-y$	0	0

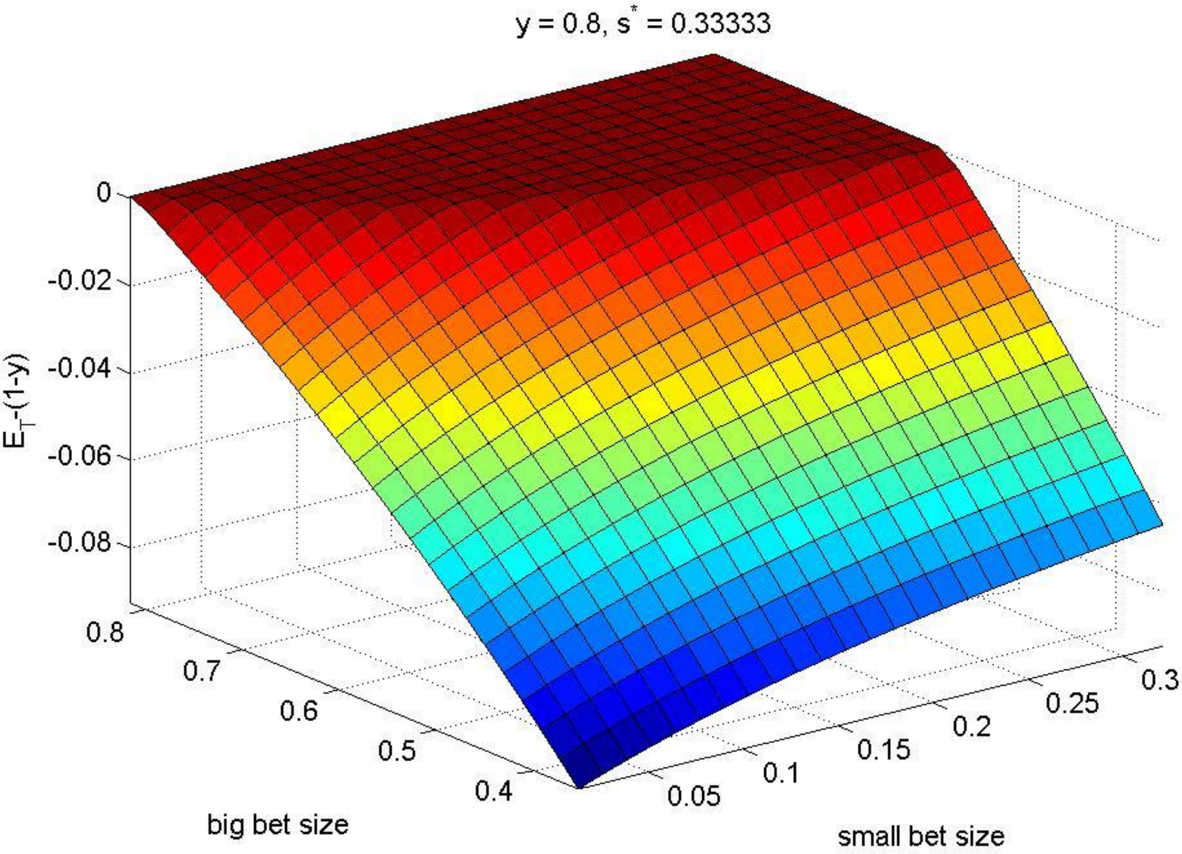
With all of these strategies available, this is a bit harder to analyse than the matrix we looked at earlier. However, if you take a look at the grey shaded cell at the bottom left, which corresponds to Tom always betting big and John always folding, you'll find that it is a Nash equilibrium. Since $s_1 > s^*$, we have $1-y < (2y-1)\,s_1$, which means that John can't reduce his loss by changing to a strategy that involves calling some bets. Similarly, Tom can't win any more than $1-y$ by using the smaller bet size or checking with air. This confirms Chen and Ankenman's verbal argument that Tom just bets big and John has to fold. However, there is more that you can say about this problem. Have a look at the four green cells. Given that two of the entries are positive (Tom wins) and two are negative (John wins) there exists a cycle, similar to the one that we talked about above, in which the two players can become trapped; Tom bets big for value and small with a bluff → John folds to big bets and calls small bets →Tom bets small for value and big with a bluff → John folds to small bets and calls big bets → …… This is more like the NLHE river spots that I talked about earlier, but how can we study this further? We could have a look at the mixed strategy equilibrium that just involves these four strategy combinations, but there is a more general approach that we can take.

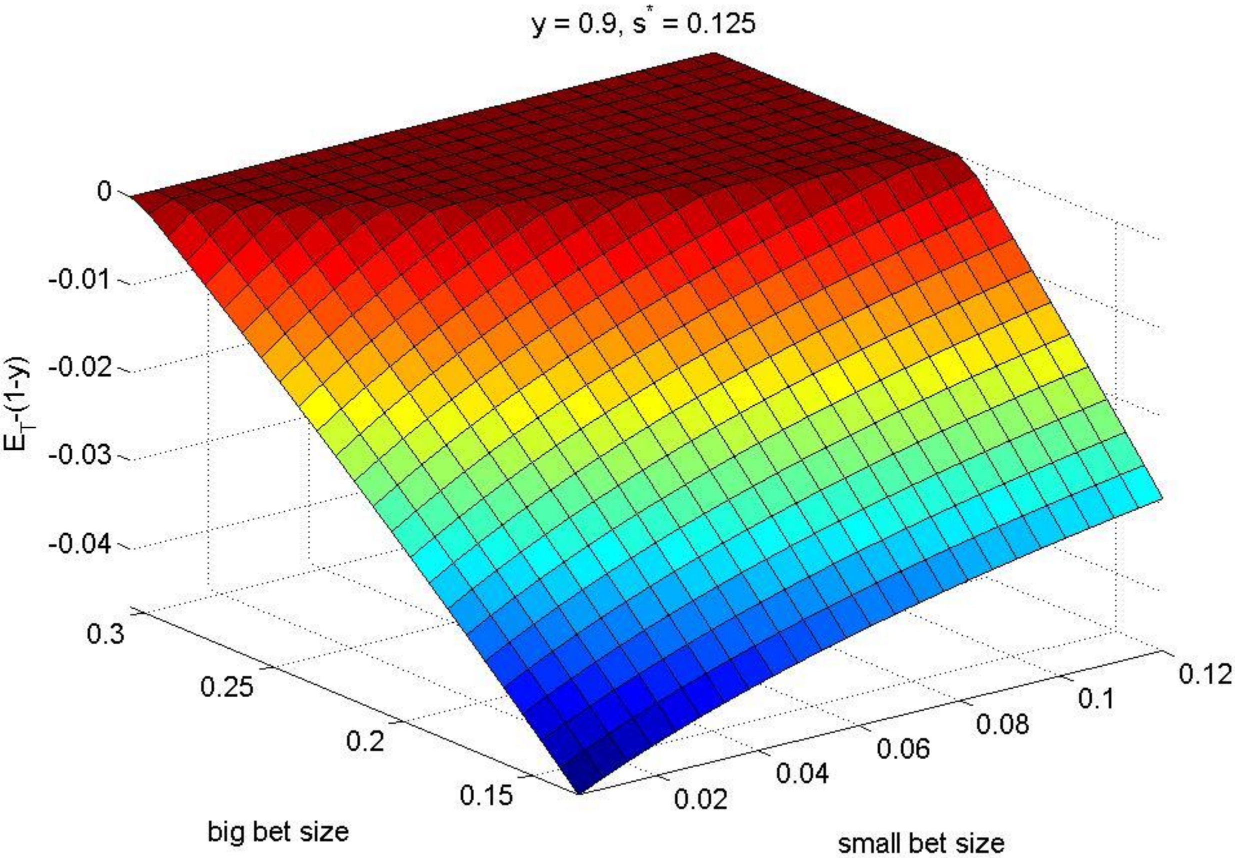
The way that I decided to proceed was to change the rules slightly so that Tom is not allowed to play the strategy v_1b_1. In other words, he can't just bet big all the time; he has sometimes to bet small or check. This removes the pure strategy Nash equilibrium, and allows us to look for a mixed strategy equilibrium instead – one that allows for the possibility of all the remaining strategies being involved. All we have to do is delete the first column from the matrix and take it from there. The matrix remains rather large so, as I'm fundamentally lazy, a trait that hasn't always been entirely helpful to me either at work or at home, I slapped the matrix onto my computer and told it to find me a mixed strategy equilibrium, using the methods that I discussed in my previous article (the simplex algorithm). Here are some results.

The four graphs below show Tom's expected win rate, minus $1 - y$. As you can see, this quantity is always either strictly negative, or zero. If it's strictly negative, it means that John loses less than $1 - y$ using this strategy, and therefore does better than he would by just folding every hand. If it's zero, it means that the best that John can do is to fold every hand and lose at the rate $1 - y$.





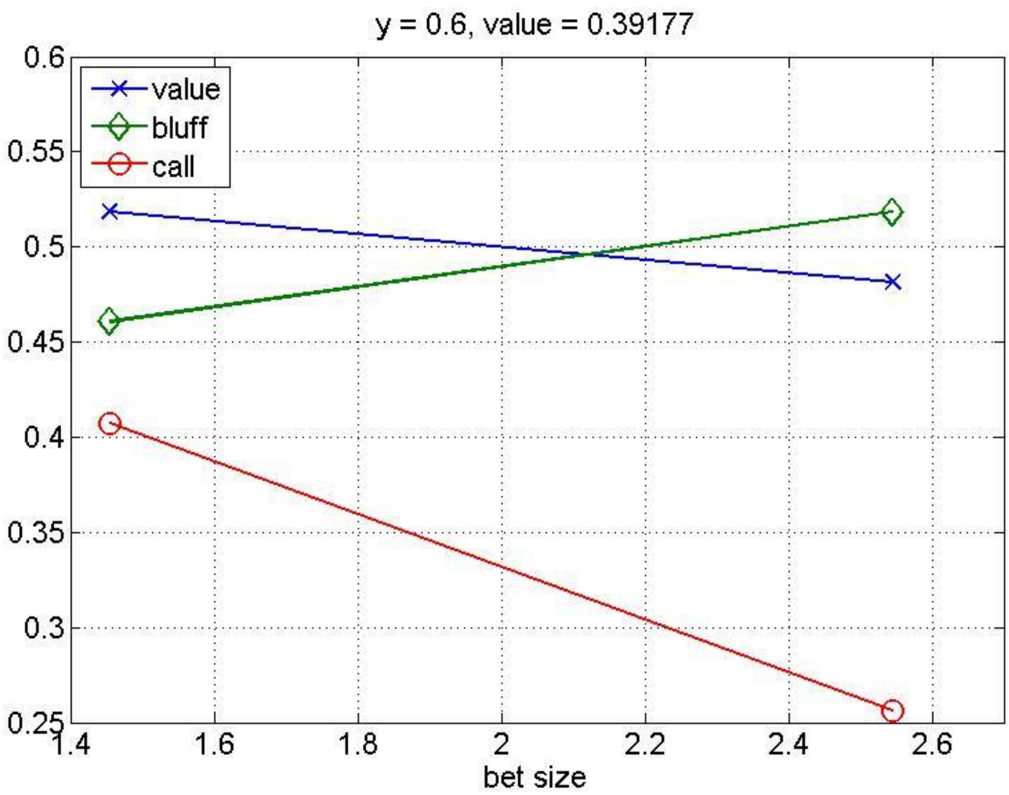

y = 0.8, s* = 0.33333
0
-0.02
-0.04
-0.06
-0.08
E_T-(1-y)
0.8
0.7
0.6
0.5
0.4
big bet size
0.05
0.1
0.15
0.2
0.25
0.3
small bet size


y = 0.9, s* = 0.125
0
-0.01
-0.02
-0.03
-0.04
E_T-(1-y)
0.3
0.25
0.2
0.15
big bet size
0.02
0.04
0.06
0.08
0.1
0.12
small bet size

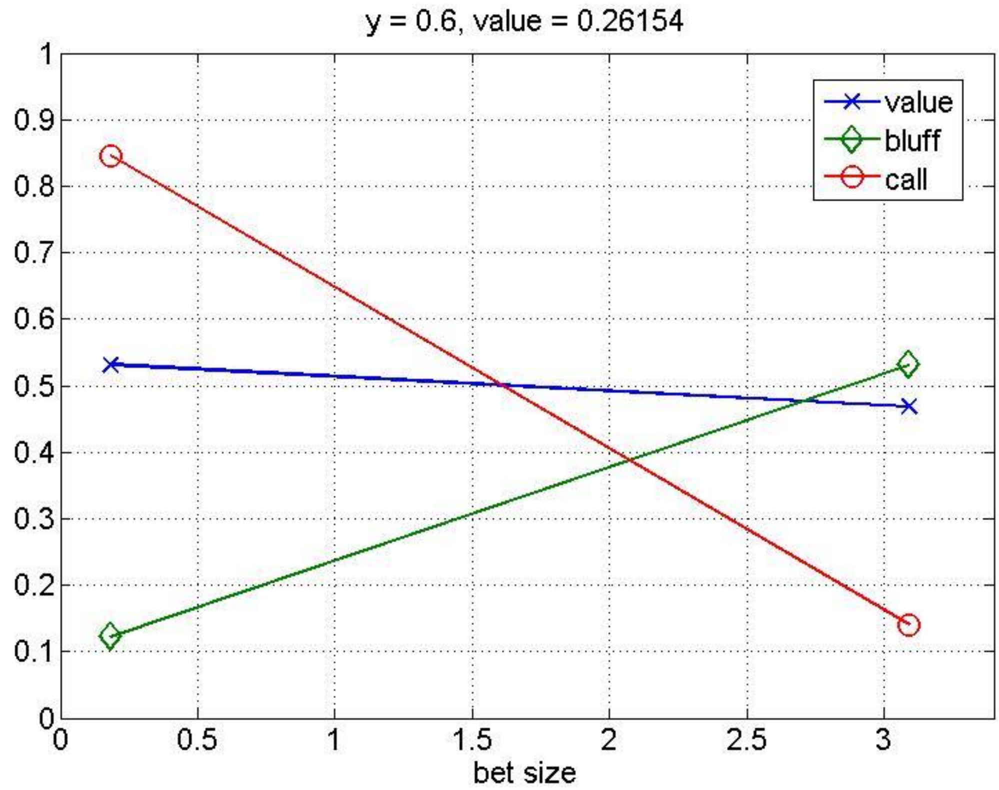
There are a number of things that we can learn from these plots:

1) If the bet sizes are small enough, John saves money by playing a mixed strategy instead of just folding all the time.
2) As y increases, and hence Tom's range gets stronger, the amount that John saves gets smaller, as does the range of bet sizes for which he can do better than just folding.
3) If we want to relate this back to NLHE, even with y up to about 0.8, the bet sizes that allow John to save money are not unreasonable for typical river situations. Beyond this, they get somewhat small, but are consistent with Tom not needing to bet much on the river to make John fold if his range is very strong.

Just to illustrate some typical mixed equilibrium strategies, here are a couple of graphs.



In the first example, Tom goes for value in roughly equal proportions between big and small bets, and similarly for bluffs. He bluffs nearly, but not quite, every time he can. John calls small bets more often than big bets and folds to a bet about 1/3 of the time. In this case, John only saves about 0.008 of a pot sized bet by using this strategy ($1 - y = 0.4$).



In the second example, which has a smaller small bet size and a bigger big bet size, Tom still bets for value in roughly equal proportions between big and small sizes, but doesn't bluff very often with a small bet, as you might expect. Note that Tom checks behind about 1/3 of the time when he has air. John will nearly always call a small bet, but folds to most big bets. He can save about 0.14 of a pot sized bet by using this strategy, which is significantly better than in the first example.

Now that we've solved this game, what does it tell us about the full HSBSCG, when Tom has the option to bet big with every hand? Well, equilibrium arguments can't tell us how the full game will play out in practice if our two protagonists can change their strategies dynamically. Let's imagine Tom starts out by relentlessly betting big, and John folds every time. What will John do if Tom now starts to throw in some small bets? Will he continue to make his disciplined folds and sit at the pure strategy Nash equilibrium? Maybe, but if he can save some money by playing a mixed strategy like one of those shown above, he may be tempted to give it a try. Perhaps Tom will let him do this for a while, but then switch back to just betting big. This is more like my experience of real poker. Analysing the dynamics of these situations is much harder than simply calculating the Nash equilibrium, both for humans and computers, as I discussed in the book.

I hope you've found this article interesting. I can now carry on with my rereading of TMOP, and hopefully I'll be able to get a bit further without being distracted.