Bonus Maths 1: From the AKQ Game to the [0, 1] Game (JB)

As well as giving our readers some extra hand examples, we've decided that I will be allowed to write some more about the mathematics of poker. I'm not going to promise that everything that I write (or indeed anything that I write), will be of direct use to you at the poker table, but I think that toy poker games are interesting, I don't have anywhere else to write about them, and it's partly my website, so you'll just have to put up with me.

The topic of my first set of ramblings is something that I touched on in the book. In the AKQ game, card removal effects couldn't really be any stronger; if you have the Q, and you'll be dealt that 1/3 of the time, you know that you have the nut low, and similarly if you have the A, you know you have the nuts. In the [0, 1] game there are no card removal effects, because there is an uncountably infinite set of possible 'hands'. The question that I want to address here is, what happens if you start adding cards to the deck? The first extension of the AKQ game is the AKQJ game, and I will talk about this in some detail below. Add another card, and we get the AKQJT game. In general, we can consider the N card game, where the deck has N cards, $C_1 < C_2 < C_3 \ldots < C_N$. In the limit $N \to \infty$, we might expect, and in fact von Neumann showed 70 years ago, that the unexploitable bluffing, value betting and calling ranges approach those of the [0,1] game, although he wasn't able to compute the unexploitable strategy for these games, as electronic computers didn't exist at the time. He did have a large role to play later on in the invention of the first computers, so maybe, if I stretch my imagination a little bit I can give poker some of the credit for precipitating the development of the computer……maybe.

We'll begin by trying to solve the AKQJ Game using the method that we used in the book. We'll succeed, but be driven to the edge of sanity doing it. We'll then have a look at a less maddening, more algorithmic method, and illustrate how to use it to solve both the AKQ and AKQJ games. Finally, we'll apply this method to the N card game and have a look at the results.

The AKQJ Game

In the book, I discussed the AKQ game with pot size \$P, and showed that John's ex-showdown expectation is

$$E(J) = \frac{1}{6}bc - \frac{1}{6}Pb(1-c) - \frac{1}{6}c + \frac{1}{6}b$$

$$= \frac{1}{6}b(P+1)\left\{c - \frac{(P-1)}{(P+1)}\right\} - \frac{1}{6}\,c = -\frac{1}{6}(P+1)c\left(\frac{1}{P+1} - b\right) - \frac{1}{6}(P-1)b,$$

where b is Tom's bluffing frequency with a Q and c John's calling/bluff catching frequency with a K. From the final two expressions above, it's fairly clear that Tom's unexploitable bluffing frequency is 1/(P+1), and John's unexploitable calling frequency is (P-1)/(P+1).

It's all very well to use this sort of 'have a look at the equations and everything will be obvious' approach to a game as simple as this, but that's not going to get us very far with more complicated games. Let's add one card to the deck and play the AKQJ game.

Tom will always value bet an A, but should he value bet a K? How often should he bluff with a J? Can he bluff with a Q?

John will always call with an A and fold a J, but how often should he bluff catch with a K and/or a Q?

Just adding one card increase the number of unknown betting frequencies from two to five.

Let's call Tom's betting frequencies with K, Q and J, b_K, b_Q and b_J, and John's calling frequencies with a K or Q c_K and c_Q. If you work your way through all the possibilities, you'll find that John's ex-showdown expectation is

$$E(J) = \frac{1}{12}\Big\{\Big((1+P)b_J - b_K - 1\Big)c_Q + \Big((1+P)b_J + (1+P)b_Q - 1\Big)c_K + (1-2P)b_J + (1-P)b_Q + b_K\Big\}$$

$$= \frac{1}{12}\Big\{\Big((1+P)(c_Q + c_K) + 1 - 2P\Big)b_J + \big((1+P)c_K + 1 - P\big)b_Q + \big(1 - c_Q\big)b_K - c_Q - c_K\Big\}.$$

Do you feel like glancing at this equation and extracting the unexploitable solution? Not so easy is it. Even when we try systematically to use the same ideas that we used for the AKQ game, we arrive at the rather bewildering

1) $c_Q + c_K = \frac{2P-1}{P+1}$, or $c_Q + c_K > \frac{2P-1}{P+1}$ and $b_J = 0$, or $c_Q + c_K < \frac{2P-1}{P+1}$ and $b_J = 1$.
2) $c_K = \frac{P-1}{P+1}$, or $c_K > \frac{P-1}{P+1}$ and $b_Q = 0$, or $c_K < \frac{P-1}{P+1}$ and $b_Q = 1$.
3) $c_Q = 1$, or $c_Q < 1$ and $b_K = 0$.
4) $(1+P)b_J - b_K = 1$, or $(1+P)b_J - b_K < 1$ and $c_Q = 0$, or $(1+P)b_J - b_K > 1$ and $c_Q = 1$.
5) $b_J + b_Q = \frac{1}{P+1}$, or $b_J + b_Q > \frac{1}{P+1}$ and $c_K = 1$, or $b_J + b_Q < \frac{1}{P+1}$ and $c_K = 0$.

It is possible to construct a logical argument based on these five statements (see the Appendix), which leads to the conclusion that the unexploitable strategy is

$$b_J = \frac{1}{P+1}, b_Q = 0, b_K = 0, c_Q = \max\left\{0, \frac{P-2}{P+1}\right\}, c_K = \min\left\{1, \frac{2P-1}{P+1}\right\},$$

provided that P≥1, which delivers Tom a profit of (2P-1)/12(P+1). That's 1/12(P+1) more than he gets from the AKQ game. Tom bluffs at the usual frequency with the bottom of his range, and value bets only with an A. John bluffcatches with Ks and Qs often enough to be indifferent to whether or not Tom bluffs. In particular, if P>2, John has to start calling with some Queens as well as all his Kings. If you glance at the appendix, you'll see that the argument is somewhat convoluted and, although it can probably improved upon, it would be hard to get a computer to construct it for you, and also hard to generalize to games with more cards. If you fancy having a go for the AKQJT, good luck to you, but I'm not up for it. We need a more efficient, more algorithmic approach.

Another Approach to Solving the AKQ Game

Let's take a step back and look at the AKQ game again. Tom has two options: he can bluff with a Q, or he can check and give up with a Q. Similarly, John has two strategic options: he can call with a K or he can fold a K. Although we have been talking about what happens if John and Tom can use a *mixed* *strategy*, i.e. a strategy that involves sometimes taking one option and sometimes the other, let's think about the payoffs if each player can only use a *pure strategy*, i.e. he has to choose one strategy or the other and stick to it. We can then construct a *payoff matrix*,

	Bluff with a Q	*Don't Bluff with a Q*
Call with a K	1/6	-1/6
Fold a K	-(P-1)/6	0

You can calculate the entries in this matrix by putting b=0 or 1 and c=0 or 1 as appropriate in the expression for E(J) that I gave earlier. John wants to choose an option which gives an entry that's as positive as possible, whilst Tom wants to choose an option that gives an entry that's as negative as possible. If Tom always bluffs, John should always call, so then Tom should never bluff, and then John should never call, which suggests that Tom should always bluff….and round and round it goes. There's no pure, equilibrium strategy, which means that the equilibrium strategy must be mixed, with the usual bluffing and calling frequencies b and c.

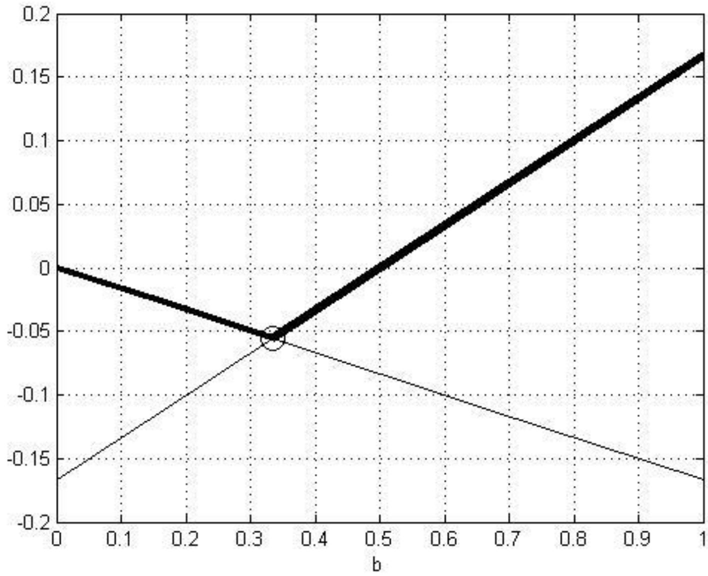
Let's now think about how we can find b and c from this matrix. Multiplying Tom's bluffing and not bluffing frequencies, b and 1-b, down the columns of the matrix gives Tom an expectation of (2b-1)/6 if John calls with a K and –(P-1)b/6 if he folds with a K. John would like to choose the strategy that gives him the largest payoff, i.e. max((2b-1)/6,-(P-1)b/6). Tom wants to minimise John's payoff, so he wants to choose b to minimise this quantity. In other words, the optimal choice of b is

$$\min_b \max\left\{\frac{1}{6}(2b-1), -\frac{1}{6}(P-1)b\right\}.$$

Tom should choose b to get the smallest value of the larger of these two functions. This is the *minimax method*, introduced by von Neumann and Morgenstern in their seminal book *The Theory of Games and Economic Behaviour*, published in 1944. It's pretty straightforward to determine b as the two functions are just straight lines. There are two cases:

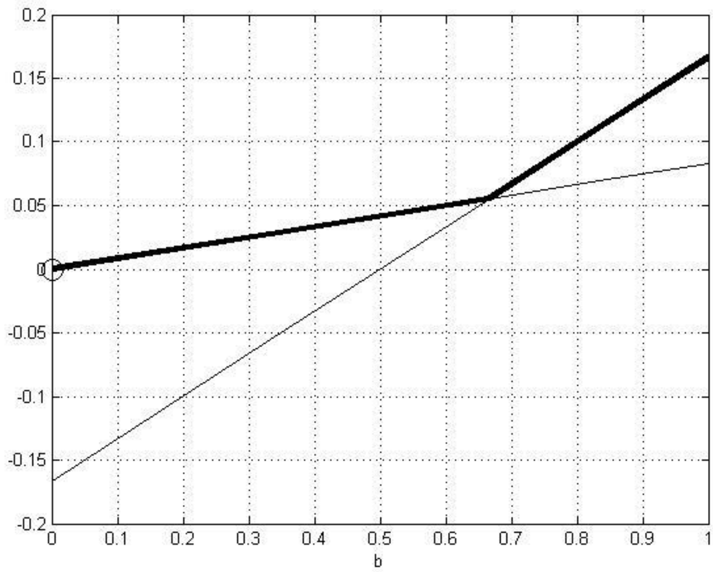
i) P>1

The two straight lines, (2b-1)/6 and –(P-1)b/6, are shown below for the typical case P=2. The thick lines show the larger of the two values for each b (the 'max' part), and the smallest value on these thick lines is circled, and occurs at the intersection of the two lines, which is, as expected, at b=1/3.



ii) P<1

The typical case P = ½ is shown below. Now that both lines have a positive slope, the smallest value on the thick lines is at b=0. For P<1, the optimal solution is for Tom never to bluff.



If that seems a pointlessly complicated and confusing way to find the optimal solution, and you may be right for the AKQ game, let's have a look at the AKQJ game, where the apparently simple approach got us into trouble that we barely managed to get out of.

Minimax for the AKQJ Game

In the AKQJ game, Tom must decide whether to bet with J, Q and K. This leads to eight pure strategic options, made up of the eight possible combinations of do/don't bet with a J, do/don't bet with a Q and do/don't bet with a K. Similarly, John must decide whether to call with his Kings and Queens, which leads to four strategic options. The payoff matrix for the AKQJ game is therefore

	(1,1,1)	(0,1,1)	(1,0,1)	(1,1,0)	(0,0,1)	(0,1,0)	(1,0,0)	(0,0,0)
(1,1)	1/4	0	1/12	1/4	-1/6	0	1/12	-1/6
(0,1)	(4-P)/12	1/6	(2-P)/12	(3-P)/12	0	1/12	(1-P)/12	-1/12
(1,0)	(1-P)/6	-P/12	(1-P)/12	(1-P)/6	-1/12	-P/12	(1-P)/12	-1/12
(0,0)	(1-P)/4	(2-P)/12	(1-P)/6	(2-3P)/12	1/12	(1-P)/12	)1-2P)/12	0

It's 4 x 8, and has eight times as many entries as the AKQ game's payoff matrix. In this table, I've denoted the strategies using ones and zeros, so that, for example, Tom's strategy of always betting with a J, but checking with a Q or K is (1,0,0), and John's strategy of calling with a K and folding a Q is (0,1). If you look at the rows for (0,1) and (1,0), you'll see that in every column the entry is larger for (0,1) than for (1,0). In other words, calling with a K and folding a Q is always better than calling with a Q and folding a K, as you might expect. We say that (1,0) is a *dominated strategy,* because (0,1) is better whatever Tom does, so that (1,0) can be discarded as an option, although I won't do that for now.

If we now assign frequencies b_i for i = 1,2,3,…8 for each of Tom's eight strategies and c_j for j = 1, 2, 3, 4 to each of John's four strategies, we can formulate a minimax problem just as we did for the AKQ game. Of course it's rather more complicated, and at first sight it would appear that we have got ourselves into as much trouble as we did with our previous attempt at solving the AKQJ game. However, it turns out that minimax problems are easy to solve using the *simplex algorithm*. The simplex algorithm is used to solve linear programming problems, of which the minimax problem is an example, and it is straightforward to program a computer to solve the problem for a given payoff matrix, although I won't go into the details here. Suffice to say that if I feed the AKQJ game's payoff matrix into my minimax solver, it spits out the correct solution.

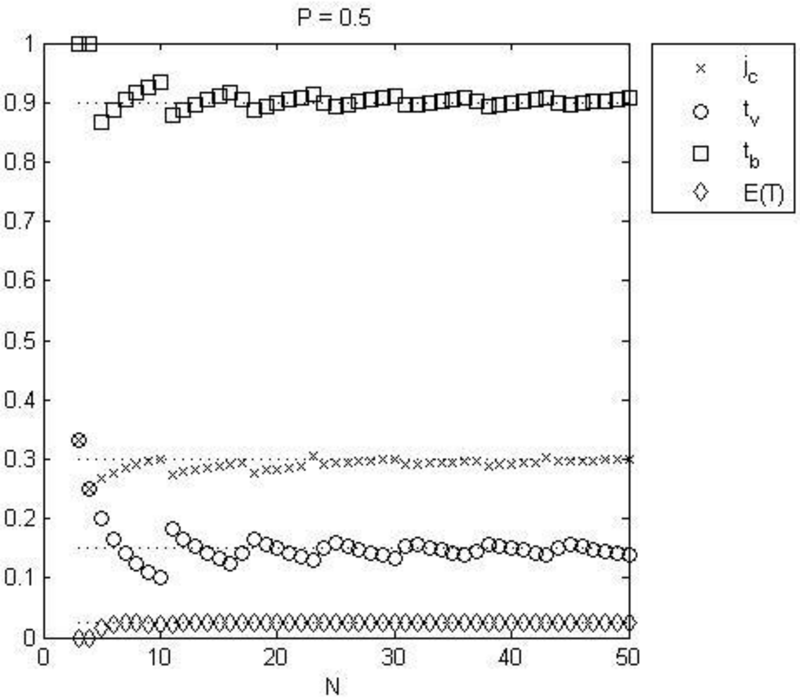
Minimax for the N Card Game

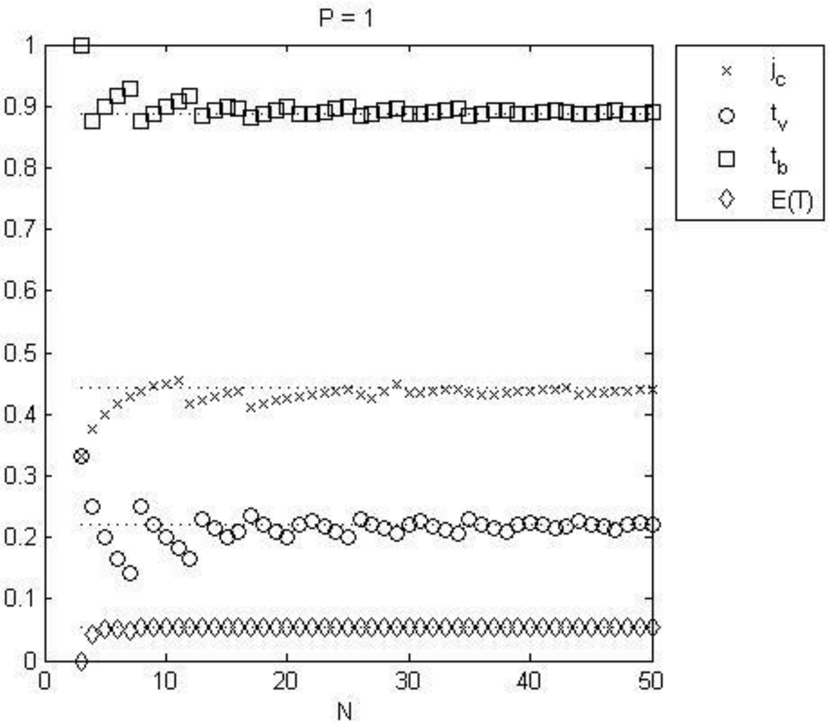
Now that we know how to solve these sorts of games algorithmically, let's attack the problem of a game with N cards, $C_1 < C_2 < C_{3\,.....} < C_N$. Here, C_1 is a the worst card in the deck and C_N is the Ace. Both players know what to do with an Ace, and John will fold the worst card, C_1, to a bet, so Tom must decide what to do with N-1 of the cards, and John with N-2 of the cards. This means that Tom has 2^{N-1} strategic options and John 2^{N-2}. Since we'd like to compute what happens as N gets large, this presents us with a technical difficulty. The number of strategic options grows exponentially with N, so that with, for example, a 100 card deck, each player has about 10^{30} strategic options, which means that the payoff matrix has about 10^{60} entries; that's 1 with 60 zeros after it. With 133 cards in the deck the payoff matrix has about 10^{80} entries in it; about one for each atom in the known universe. I'm pretty sure that either of these matrices would be tricky to store in my laptop. Fortunately there are lots of clever methods to get around this problem, and, since I'm a novice at game theory, I'm not going to claim to know much about them.

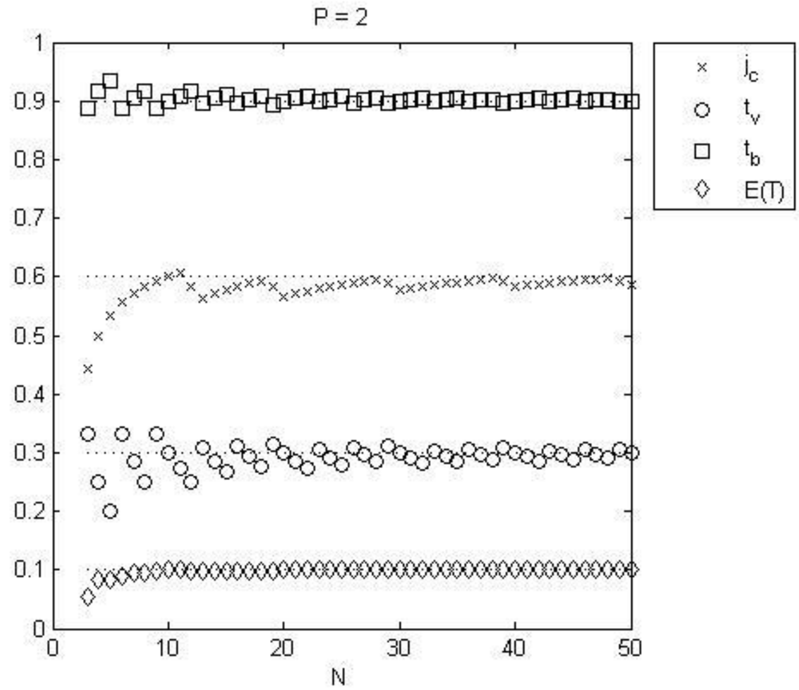
Fortunately, there is a simple way to cut down the size of our computational problem. Guided by what we know about the solution of the AKQ, AKQJ and [0,1] games, we can be confident that the optimal strategy for Tom is to value bet with the top of his range, bluff with the bottom of his range and check behind with his midstrength hands, and that John should call with the top of his range and fold the bottom of his range. The only thing that we need to calculate is how wide these value betting, bluffing and calling ranges should be. This means that we only need to consider strategies of the form (1,1,…..1,1,0,0….,0,0,…1,1,…,1,1) for Tom and (0,0,…0,0,1,1,…1,1) for John. If you count these up, that's 1+N(N-1)/2 strategies for Tom and N-1 strategies for John, so that the payoff matrix has (N-1)(1+N(N-1)/2) entries. The size therefore grows like $N^3/2$ for N large, and the matrix can get quite big, but not so big that we can't solve the problem up to N=100, and probably somewhat larger than that, without breaking too much computational sweat. For example, my bog standard desktop PC takes about three minutes to find the optimal strategy when N=100.

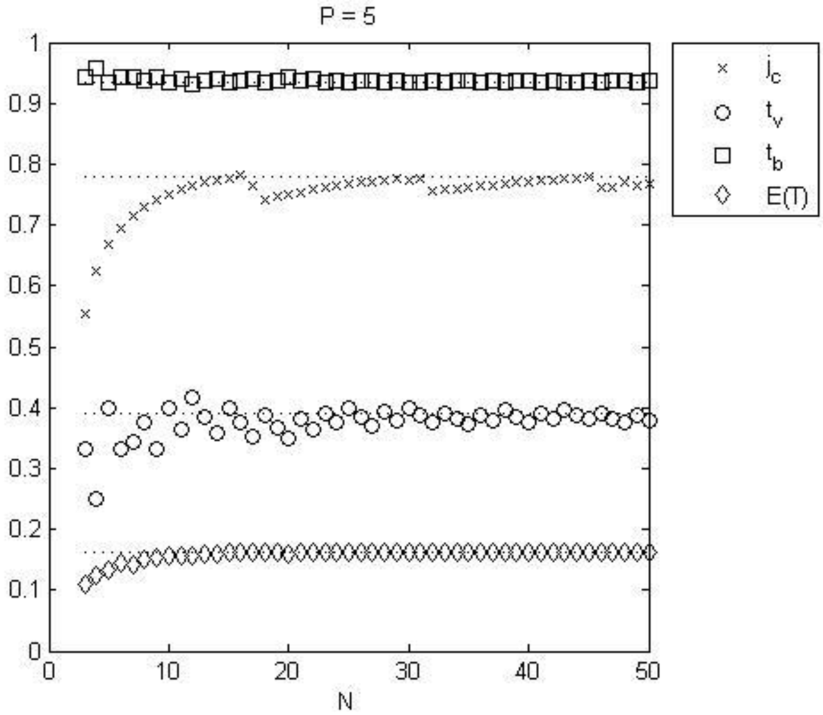
The four pictures below show the unexploitable strategies for decks with up to 50 cards for various pot sizes, P. In each case, j_c is John's unexploitable calling range, t_v Tom's unexploitable value betting range, and t_b his unexploitable bluffing range. The dotted lines give these ranges for the [0, 1] game, and we can see that as N gets larger these ranges do indeed agree very well. Although the ranges can change dramatically with the addition of one extra card, at least for a small deck, Tom's expected win rate changes quite smoothly and converges rapidly to the value predicted for the [0,1] game. In each case illustrated, once the deck has about eight cards, the winrate predicted by the [0, 1] game is a good approximation, even though the unexploitable ranges predicted by the [0, 1] game are not.

Whether these results have any relevance to real poker is not clear. Remember that, if we want to draw an analogy between these simple games and NLHE river situations, we need to think of each of the 'cards' in the C_N game's deck as analogous buckets of NLHE hands. For example, a river situation where Tom has an equal number of combos of the nuts and air and John has a bluff catcher is more closely related to the AKQ game (N=3) than to a game with a larger value of N. It may well be that the results I've discussed here aren't directly relevant to any real poker variant, but they do provide a nice introduction to some of the ideas of Game Theory.


P = 0.5
N
j_c
t_v
t_b
E(T)


P = 1
N
j_c
t_v
t_b
E(T)


P = 2
N
j_c
t_v
t_b
E(T)


P = 5
N
j_c
t_v
t_b
E(T)

Appendix 1: Solving the AKQJ Game

Hold onto your hats - we're going to be trying to find the solution of

1) $c_Q + c_K = \frac{2P-1}{P+1}$, or $c_Q + c_K > \frac{2P-1}{P+1}$ and $b_J = 0$, or $c_Q + c_K < \frac{2P-1}{P+1}$ and $b_J = 1$.
2) $c_K = \frac{P-1}{P+1}$, or $c_K > \frac{P-1}{P+1}$ and $b_Q = 0$, or $c_K < \frac{P-1}{P+1}$ and $b_Q = 1$.
3) $c_Q = 1$, or $c_Q < 1$ and $b_K = 0$.
4) $(1+P)b_J - b_K = 1$, or $(1+P)b_J - b_K < 1$ and $c_Q = 0$, or $(1+P)b_J - b_K > 1$ and $c_Q = 1$.
5) $b_J + b_Q = \frac{1}{P+1}$, or $b_J + b_Q > \frac{1}{P+1}$ and $c_K = 1$, or $b_J + b_Q < \frac{1}{P+1}$ and $c_K = 0$.

We'll assume that P≥1, just to make things a bit easier. Let's start by considering the three possibilities in case 1).

i) $b_J = 1$: Then 4) implies that $c_Q = 1$ and 5) implies that $c_K = 1$, so that 1) implies that $b_J = 0$, which is a contradiction.

ii) $b_J = 0$: Then 4) implies that $c_Q = 0$ and 3) implies that $b_K = 0$. Then 2) and 5) become

$c_K = \frac{P-1}{P+1}$, or $c_K > \frac{P-1}{P+1}$ and $b_Q = 0$, or $c_K < \frac{P-1}{P+1}$ and $b_Q = 1$.
$b_Q = \frac{1}{P+1}$, or $b_Q > \frac{1}{P+1}$ and $c_K = 1$, or $b_Q < \frac{1}{P+1}$ and $c_K = 0$.

The only consistent solution of these is

$b_Q = \frac{1}{P+1}, c_K = \frac{P-1}{P+1}$, so that $c_Q + c_K = \frac{P-1}{P+1}$,

and then 1) implies that) $b_J = 1$, which is a contradiction.

iii) $c_Q + c_K = \frac{2P-1}{P+1}$: We now need to consider the two subcases from 3)

a) $c_Q = 1$: Then $c_K = \frac{P-2}{P+1}$, and 2) implies that b_Q =1 and 5) implies that $c_K = 1$, which is a contradiction.

b) $b_K = 0$ and $c_Q < 1$: Next there are two subcases arising from 4), which now reads

$b_J = \frac{1}{P+1}$, or $b_J < \frac{1}{P+1}$ and $c_Q = 0$,

after eliminating the third possibility, which does not have $c_Q < 1$.

I) $b_J < \frac{1}{P+1}$ and $c_Q = 0$: This means that $c_K = \frac{2P-1}{P+1}$. Then 2) implies that $b_Q = 0$ and 5) implies that $c_K = 0$, a contradiction.

II) $b_J = \frac{1}{P+1}$: Then 5) is reduced to just two possibilities, namely $b_Q = 0$, or $b_Q > 0$ and $c_K = 1$, and then 2) shows that $b_Q = 0$.

So, after all that, we find that in order to satisfy all five equations, we need

$$b_J = \frac{1}{P+1}, b_Q = b_K = 0, c_Q + c_K = \frac{2P-1}{P+1}.$$

As you can see, the solution is not fully determined. As long as John calls at the correct frequency with Kings or Queens combined, he is unexploitable. However, he doesn't maximize his winrate if Tom (exploitably) bets with a K or Q himself. John should obviously prefer calling with Kings to calling with Queens, and this leads to the optimal solution

$$c_Q = \max\left\{0, \frac{P-2}{P+1}\right\}, c_K = \min\left\{1, \frac{2P-1}{P+1}\right\}.$$

This indeterminacy persists when solving the N card game, and needs to be taken into account when presenting the optimal solution calculated using the simplex algorithm.